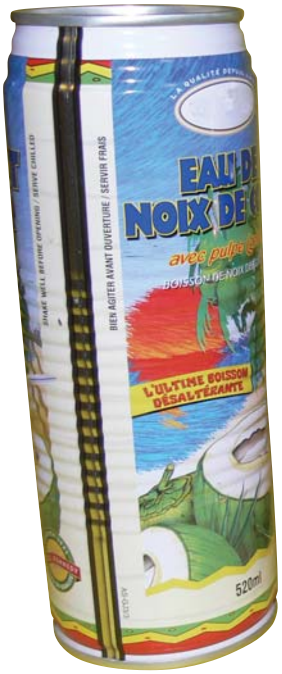
9.5 Maximize the Volume of a Cylinder

The National Packaging Competition is sponsored by the Packaging Association of Canada. This competition is held every 2 years to promote the Canadian packaging and design industries. One of the packaging categories at the competition is rigid and semi-rigid containers. Entries are judged on creativity, effectiveness of communication, originality, environmental considerations, and product suitability.



In Section 9.4, you learned how to maximize the volume of a square-based prism for a given surface area. In this section, you will perform the same investigation for a cylinder.

Investigate

How can you compare the volumes of cylinders with the same surface area?

Method 1: Use a Table

Your task is to design a cylindrical juice can that uses no more than 375 cm^2 of aluminum. The can should have the greatest capacity possible.

1. To investigate the volume of the cylinder as its radius changes, you will need an expression for the height in terms of the radius.
- Start with the formula for the surface area of a cylinder: $SA = 2\pi r^2 + 2\pi rh$.
- Substitute 375 cm^2 into the formula.
- Rearrange the formula to express the height, h, in terms of the radius, r.

Copy the steps and write a short explanation beside each. The first has been done for you.

Step	***Explanation***
$SA = 2\pi r^2 + 2\pi rh$	
$375 = 2\pi r^2 + 2\pi rh$	**Substitute *SA* = 375.**
$375 - 2\pi r^2 = 2\pi r^2 + 2\pi rh - 2\pi r^2$	
$375 - 2\pi r^2 = 2\pi rh$	
$\dfrac{375 - 2\pi r^2}{2\pi r} = \dfrac{\overset{1}{\cancel{2\pi r}}h}{\underset{1}{\cancel{2\pi r}}}$	
$h = \dfrac{375 - 2\pi r^2}{2\pi r}$	

Making Connections

You learned how to rearrange formulas in 4.4 Modelling With Formulas.

2. Let the radius be 1 cm.

a) Determine the height of the can by using the algebraic model you found in step 1.

b) Determine the volume of this can using the formula for the volume of a cylinder: $V = \pi r^2 h$. Record the data in a table.

Radius (cm)	Height (cm)	Volume (cm^3)	Surface Area (cm^2)
1			375
2			375
3			375

3. Repeat step 2, letting the radius take whole-number values from 2 cm to 7 cm.

4. What is the maximum volume for the cans in your table? What are the radius and height of the can with this volume?

5. Reflect Do these dimensions give the optimal volume for the surface area of 375 cm^2? Describe how you could extend your investigation to determine the dimensions of a can with a volume greater than the value in the table.

Method 2: Use a Spreadsheet

Use a spreadsheet to investigate the volume of a cylinder with a surface area of 375 cm^2 as the radius changes. The spreadsheet will let you investigate values for the radius that are not whole numbers.

1. Create a spreadsheet with formulas as follows. Notice that the formulas for height and volume are the same as those used in Method 1.

	A	B	C	D
1	Radius (cm)	Height (cm)	Volume (cm^3)	Surface Area (cm^2)
2	1	=(375-2*PI()*A2^2)/(2*PI()*A2)	=PI()*A2^2*B2	375
3	2	=(375-2*PI()*A3^2)/(2*PI()*A3)	=PI()*A3^2*B3	375

2. Use **Fill Down** to complete the spreadsheet for radius values from 1 cm to 7 cm. What happens when you enter a radius value greater than 7 cm? Explain why this happens.

3. What is the whole-number radius value of the cylinder with the greatest volume? Try entering a radius value 0.1 cm greater than this value. Does the volume increase? If not, try a value 0.1 cm less. Continue investigating until the volume is a maximum for the radius value, to the nearest tenth of a centimetre.

4. Keep refining the radius value to hundredths of a centimetre. The volume should be slightly greater than your last attempt.

5. a) Record the radius and height of the can with the optimal volume.
   b) How do the values of the radius and height of this can compare?
   c) How do the values of the diameter and height of the can compare?

6. Change the formulas in the spreadsheet to investigate the dimensions of a cylinder with maximum volume if the surface area is 500 cm^2. How do the radius and height compare?

7. Repeat step 6 for a cylinder with a surface area of 1000 cm^2.

8. **Reflect**
   a) Describe any relationship you notice between the radius and height of a cylinder with maximum volume for a fixed surface area.
   b) How does this compare to the relationship between the dimensions of a square-based prism?

9. Save this spreadsheet for use in later investigations.

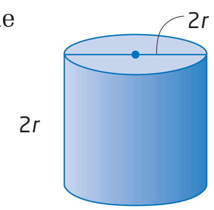
Example Maximize the Volume of a Cylinder

a) Determine the dimensions of the cylinder with maximum volume that can be made with 600 cm^2 of aluminum. Round the dimensions to the nearest hundredth of a centimetre.

b) What is the volume of this cylinder, to the nearest cubic centimetre?

Solution

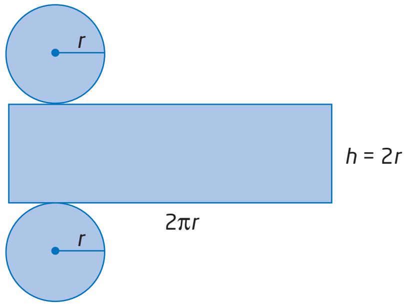
a) For a given surface area, the cylinder with maximum volume has a height equal to its diameter.



If I look at this cylinder from the front, it looks like a square.

Substitute $h = 2r$ into the formula for the surface area of a cylinder.

$$\begin{aligned} SA &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r^2 + 2\pi r(2r) \\ &= 2\pi r^2 + 4\pi r^2 \\ &= 6\pi r^2 \end{aligned}$$



Substitute the surface area of 600 cm^2 to find the dimensions of the cylinder.

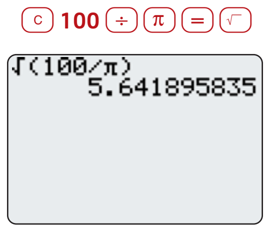
$$600 = 6\pi r^2$$

$$\frac{\overset{100}{\cancel{600}}}{\underset{1}{\cancel{6\pi}}} = \frac{\overset{1}{\cancel{6\pi}} r^2}{\underset{1}{\cancel{6\pi}}}$$ **Divide both sides by 6π.**

$$\frac{100}{\pi} = r^2$$

$$\sqrt{\frac{100}{\pi}} = r$$ **Take the square root of both sides.**

$$5.64 \doteq r$$



The radius of the cylinder should be 5.64 cm and the height should be twice that, or 11.28 cm.

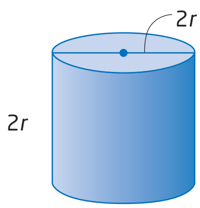
b) Use the formula for the volume of a cylinder.

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(5.64)^2(11.28) \\ &\doteq 1127 \end{aligned}$$ **Estimate: $3 \times 6^2 \times 11 = 1188$**

The volume of this cylinder is about 1127 cm^3.

Key Concepts

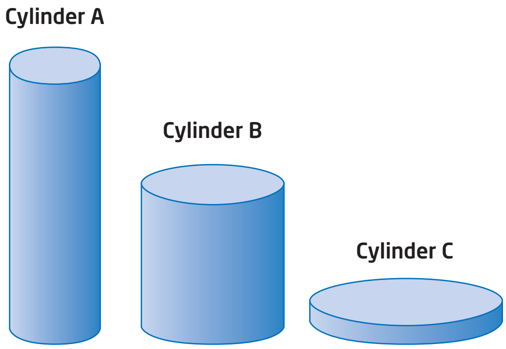
- For a cylinder with a given surface area, a radius and a height exist that produce the maximum volume.
- The maximum volume for a given surface area of a cylinder occurs when its height equals its diameter. That is, $h = d$ or $h = 2r$.
- The dimensions of the cylinder with maximum volume for a given surface area can be found by solving the formula $SA = 6\pi r^2$ for r, and the height will be twice that value, or $2r$.



Communicate Your Understanding

C1 Describe a situation where it would be necessary to find the maximum volume of a cylinder, given its surface area.

C2 These cylinders have the same surface area. Which cylinder has the greatest volume? Explain your answer.



C3 Not all drinking glasses are designed to have the greatest volume for a given surface area. Why might these glasses be designed in other ways?

Practise

For help with questions 1 and 2, see the Example.

1. Determine the dimensions of the cylinder with the maximum volume for each surface area. Round the dimensions to the nearest hundredth of a unit.

a) 1200 cm^2 **b)** 10 m^2

c) 125 cm^2 **d)** 6400 mm^2

2. Determine the volume of each cylinder in question 1. Round to the nearest cubic unit.

Connect and Apply

3. Many European businesses buy aircraft manufactured in North America. To make the flight home across the Atlantic Ocean, extra fuel tanks are often carried in the cabin of the plane. These extra fuel tanks, called ferry tanks, must be as light as possible. A cylindrical ferry tank is to be made from 8 m^2 of aluminum. What is the maximum volume of fuel that it can hold, to the nearest cubic metre?

Did You Know?

The first trans-Atlantic crossing by a miniature robotic airplane occurred on August 25, 1998. The unpiloted airplane took 26 h to make the over 3200 km flight, taking off from Bell Island, Newfoundland and Labrador, and landing in the Hebrides Islands of Scotland.

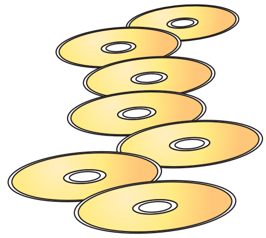
4. A fertilizer company wants to make a cylindrical storage container of sheet metal. 72 m^2 of metal is available.

a) Determine the dimensions of the container with maximum volume. Round the dimensions to the nearest tenth of a metre.

b) Determine how many litres of liquid fertilizer this container can hold. Hint: 1 m^3 = 1000 L.

c) Describe any assumptions you have made in solving this problem.

5. **Chapter Problem** Talia ships CDs to her customers in cylindrical plastic containers. The CDs are 12 cm in diameter and 2 mm thick. Talia wants the cylinder to hold as many CDs as possible, but to use as little plastic as possible.



   a) What is the height of the optimal cylinder?

   b) How many CDs will this cylinder hold?

   c) Describe any assumptions you have made.

6. An open-topped cylinder is to be made using 500 cm^2 of plastic.

   a) Describe how you would determine the dimensions of the cylinder of maximum volume.

   b) Determine the dimensions of the cylinder with the optimal volume. Round to the nearest tenth of a centimetre.

Extend

7. You have a piece of sheet metal. Your task is to use this material to create a fuel container with maximum volume.

   a) Which shape would have the greatest volume: a square-based prism or a cylinder?

   b) Justify your answer using a fixed surface area of 2400 cm^2.

8. Suppose you have 2000 cm^2 of material to create a three-dimensional figure with the greatest volume. The material can be formed into a square-based prism, a cylinder, or a sphere.

   a) Predict which shape will produce the greatest volume.

   b) Determine the dimensions of each shape so that the volume is maximized.

   c) Determine the volume of each shape.

   d) Was your prediction correct? If not, which of the three shapes has the greatest volume for a given surface area? Will this always be true?

   e) Summarize your findings.

Making Connections

You worked with the volume and surface area formulas for a sphere in Chapter 8: Measurement Relationships:

$V = \frac{4}{3}\pi r^3$ and $SA = 4\pi r^2$

9. **Use Technology** You are to construct a cylinder that has a surface area of 2 m^2. Use a spreadsheet to investigate the dimensions of the cylinder with the greatest volume if

   a) the cylinder has a top and a bottom

   b) the cylinder has no top

10. **Math Contest** Determine the dimensions of the cylinder of maximum volume that can be inscribed in a sphere of radius 8 cm.